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Entropy as a function of geometric phase

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Abstract
We give a closed-form solution of von Neumann entropy as a function of geometric phase modulated by visibility and average distinguishability in Hilbert spaces of two and three dimensions. We show that the same type of dependence also exists in higher dimensions albeit with other terms. For non-maximal mixing, the results become more involved and generally depend also on the probability of the states. We also outline a method for measuring both the entropy and the phase experimentally using a simple Mach–Zehnder-type interferometer which explains physically why the two concepts are related.

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1. Introduction

The von Neumann entropy [1] is a measure of mixedness in a physical state described by a density matrix. The general rule is that the more orthogonal the states comprising the density matrix are, the higher the value of the corresponding entropy. Looking at it from a different perspective, the entropy signifies the lack of knowledge we have about the exact pure state the system is in. For pure states, the knowledge is maximal and the value of entropy is zero, while for a maximally mixed state (the normalized identity matrix), the value of entropy is highest as any of the pure states in the mixture is equally likely. Therefore, this intuition would suggest that distinguishability between states is the only parameter determining the value of entropy. We also note that entropy is a static property of the system (i.e., it is only a function of the density matrix describing the state, rendering it completely insensitive to the dynamical or kinematical evolution). A very readable account of the general properties of entropy can be found in Wehrl [2].

Geometric phases, on the other hand, are obtained when a physical system evolves through a (discrete or continuous) set of states. We can say that this phase depends only on the geometric aspects of this evolution (i.e., it is, for instance, independent of the rate of evolution, or more generally it is independent of the Hamiltonian that has evolved the system), but experimentally it is generated by dynamics of either a continuous Schrödinger-type evolution or a discrete quantum measurement (of the most general type). Due to the independence from the Hamiltonian, the geometric phase can be studied by considering the kinematics, i.e., the states that the system traverses. A very thorough exposition of this procedure can be found in [3, 4]. The geometric phase was originally discovered by Berry [5] who considered a cyclic adiabatic evolution of a pure state. Subsequently, this has been generalized to the nonadiabatic [6], noncyclic [7] and the mixed state [8, 9] cases. The geometric phase has a long and interesting history, and we refer the interested reader to the collection of papers compiled by Shapere and Wilczek [10]. No detailed knowledge of this will be necessary however, as all the relevant information will be given here.

Given that the entropy is a static property and geometric phase a kinematic property of a quantum system, we would not at first sight expect there to be any connections between the two. This conclusion is however incorrect and in this paper, we will show that entropy can in fact be written as a function of geometric phase (and some other parameters in general).

We offer an explanation to the relationship between entropy and geometric phase. Throughout the paper, we will be considering a density matrix consisting of discrete states $\rho = \frac{1}{N}\sum_{i=1}^{N} |\psi_i\rangle\langle\psi_i|$ where N is the number of states and $|\psi_i\rangle$ is the ith state in the ensemble. The density matrix in this way appears to be a static quantity. However, we can also think of density matrices as time averages $\rho = \frac{1}{T}\sum_{i=1}^{T} |\psi(i\,\Delta t)\rangle\langle\psi(i\,\Delta t)|$ where T is the total evolution time and $|\psi(i\,\Delta t)\rangle$ is the state at time $i\,\Delta t$. We can now think of a single state evolving from one state to another every step Δt. If we set $T = N$ and $|\psi_i\rangle = |\psi(i\,\Delta t)\rangle$, the above two density matrices are equivalent. Now that we have introduced a kinematical way of looking at the density matrix, perhaps the result of expressing the von Neumann entropy (a function of the density matrix) in terms of the geometric phase is not so unexpected.

Our work has been stimulated by Jozsa and Schlienz [11] who pointed out that von Neumann entropy can increase even when the ensemble of quantum states become less distinguishable (i.e., more parallel). They noted that this behaviour does not occur in a two-dimensional Hilbert space but emerges in a three-dimensional Hilbert space. Their conclusion is that distinguishability is a global property (considering the whole ensemble) which cannot be reduced to considering the pairwise overlaps of the states.

In this paper, we attribute this transition to the presence of a geometric phase by giving a closed-form solution of entropy as a function of geometric phase for the maximally mixed case. We will begin by defining all the relevant variables. Then we will work through the two- and three-dimensional cases. We also comment on the arbitrary dimensional case with any number of states. We will finally discuss a method to experimentally measure entropy and show that the same set-up is also used for measuring geometric phases. This gives a strong reason for why the two concepts are related. Interestingly, in two dimensions, the entropy is either a function of the phase or distinguishability, but we do not need both at the same time (this is because the phase and distinguishability can directly be related to each other). For higher dimensions this relationship becomes more complicated as we will show and throughout the paper we discuss mathematical and physical reasons for this difference between two and higher dimensional systems. We will conclude with a summary.

2. Setting the scene

As we have already said, entropy is a physical quantity that quantifies the lack of information in a given ensemble. Suppose the ensemble contains three quantum states $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$ with probabilities p_1, p_2, p_3, respectively, where $p_1 + p_2 + p_3 = 1$. We can construct the density operator $\rho = p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2| + p_3|\psi_3\rangle\langle\psi_3|$ and the von Neumann entropy as $S_{\rm vN} = -{\rm Tr}(\rho \ln \rho)$ where the Boltzmann constant $k_B = 1$. Together with state vectors, we shall also be working with coherence vectors because we can generalize easily to higher number of states than the dimension of the system. Any density operator for two dimensions (i.e., two-level system) can be written as $\rho = \frac{1}{2}(\mathbb{I} + \mathbf{n}\cdot\sigma)$, where σ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

$\mathbf{n}$ is a three-component coherence vector and $\cdot$ denotes the scalar product. In three dimensions any state can be written as $\rho = \frac{1}{3}(\mathbb{I} + \sqrt{3}\mathbf{n}\cdot\lambda)$ where λ are the Gell–Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{2}$$

and $\mathbf{n}$ is now an eight-component coherence vector. Note that our representation of the state in terms of Pauli and Gell–Mann matrices is not unique. Any other appropriate basis will be related to this basis through an orthogonal matrix transformation that would be three and eight-dimensional, respectively [12]. Here, the dimension refers to the number of real parameters needed to describe the state. For d-level systems, the density matrix is given by

$$\rho = \frac{1}{d}\left(\mathbb{I} + \sqrt{\frac{d(d-1)}{2}}\mathbf{n}\cdot\lambda\right) \tag{3}$$

where $\mathbf{n}$ is the $d^2 - 1$ element coherence vector and λ are $d \times d$ matrices satisfying the Lie algebra of $SU(d)$ [13]. The mixedness is introduced in the coherence vectors by $\mathbf{n} = p_1\mathbf{n}_1 + p_2\mathbf{n}_2 + p_3\mathbf{n}_3$ where $\mathbf{n}_i$ are the coherence vectors corresponding to the ith state.

We now introduce a quantity called the perimeter, P, defined as

$$P = |\mathbf{n}_1 - \mathbf{n}_2|^2 + |\mathbf{n}_2 - \mathbf{n}_3|^2 + |\mathbf{n}_3 - \mathbf{n}_1|^2 \tag{4}$$
$$= 2\mathbf{n}_1^2 + 2\mathbf{n}_2^2 + 2\mathbf{n}_3^2 - 2\mathbf{n}_1\cdot\mathbf{n}_2 - 2\mathbf{n}_2\cdot\mathbf{n}_3 - 2\mathbf{n}_3\cdot\mathbf{n}_1 \tag{5}$$
$$= 6 - 2(\mathbf{n}_1\cdot\mathbf{n}_2 + \mathbf{n}_2\cdot\mathbf{n}_3 + \mathbf{n}_3\cdot\mathbf{n}_1). \tag{6}$$

This quantity tells us how different the three states are on average. The larger the perimeter, the more orthogonal the states become. Note that this quantity is related to the sum of the overlaps of the quantum states (for example in three dimensions):

$$Q = |\langle\psi_1|\psi_2\rangle|^2 + |\langle\psi_2|\psi_3\rangle|^2 + |\langle\psi_3|\psi_1\rangle|^2 \tag{7}$$
$$= {\rm Tr}(\rho_1\rho_2) + {\rm Tr}(\rho_2\rho_3) + {\rm Tr}(\rho_3\rho_1) \tag{8}$$
$$= \tfrac{1}{3}(1 + 2\mathbf{n}_1\cdot\mathbf{n}_2) + \tfrac{1}{3}(1 + 2\mathbf{n}_2\cdot\mathbf{n}_3) + \tfrac{1}{3}(1 + 2\mathbf{n}_3\cdot\mathbf{n}_1) \tag{9}$$
$$= 1 + \frac{6-P}{3}. \tag{10}$$

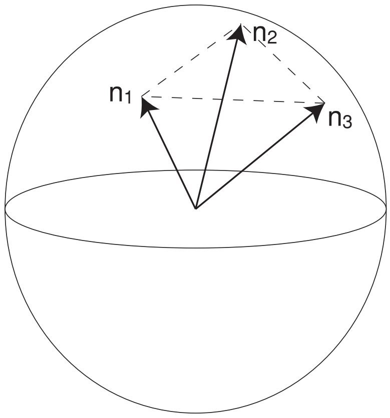

Figure 1. Coherence vector space with three general states. The dashed lines denote the perimeter. Note that in two dimensions, the ball itself is a Bloch sphere and every point corresponds to a physical state. But for higher dimensions, the coherence vector space is a proper subset of the ball [12].

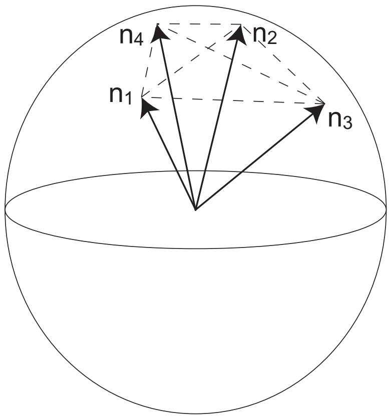

Figure 2. Coherence vector space with four states. The dashed lines again denote the perimeter.

The negative sign makes sense because the more/less parallel the states are, the smaller/larger the perimeter. If the states are identical, $P = 0$ (since $\mathbf{n}_i \cdot \mathbf{n}_i = 1$) and if they are orthogonal, $P = 9$ (since $\mathbf{n}_i \cdot \mathbf{n}_j = -1/(d-1)$ for $i \neq j$ where d is the dimension of the system). Note that this is for the three-dimensional case. With three states, we can visualize the perimeter as the square distances of each side of a triangle with each vertex representing a quantum state (see figure 1). As soon as we consider more states, the usual meaning of perimeter breaks down because we must include more than two distances for each state. For example with four states, we will have the square distances of each side of a four-sided polygon as well as the two lines adjoining opposite vertices (see figure 2). Hence, the term 'average distinguishability' may be more appropriate than perimeter but we will continue to use the latter throughout the paper. We would now expect, as mentioned earlier in the introduction, that the larger the perimeter, the more distinguishable (orthogonal) the states comprising the mixture, and the higher the value of the entropy. This is, as will be shown in more detail soon, true for qubits, but fails in higher dimensions in general.

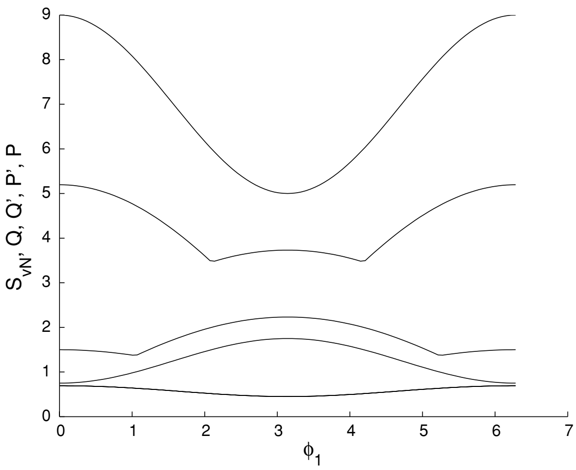

Figure 3. In accord with the order of the labels on the *Y*-axis, the bottom line shows the von Neumann entropy, the one above shows Q, then Q', P' and the top line shows P. P' and Q' are without the squares in equations (4) and (7), respectively. Note that just less than $\phi_1 = \pi$, S_{vN} decreases as P' increases. Also around $\phi_1 = 0$ and $\phi_1 = 2\pi$, S_{vN} decreases as Q' decreases and S_{vN} increases as Q' increases, respectively. Therefore, P' and Q' are counterintuitive whereas P and Q show what we expect.

Before we go into the main topic of the paper, we point out another important issue. Namely, changing the definition of the perimeter and Q by removing the squares in equations (4) and (7), respectively, changes the behaviour of the perimeter with respect to the entropy. In particular, we can now observe an increase in entropy by decreasing the perimeter (or equivalently increasing the overlap) for the two-dimensional ensemble contrary to [11]. Let us consider the following states:

$$|\psi_1\rangle = \cos(\theta_1/2)|0\rangle + \exp(-\mathrm{i}\phi_1)\sin(\theta_1/2)|1\rangle \tag{11}$$

$$|\psi_2\rangle = \cos(\theta_2/2)|0\rangle + \exp(-\mathrm{i}\phi_2)\sin(\theta_2/2)|1\rangle \tag{12}$$

$$|\psi_3\rangle = \cos(\theta_3/2)|0\rangle + \exp(-\mathrm{i}\phi_3)\sin(\theta_3/2)|1\rangle \tag{13}$$

or equivalently the following coherence vectors:

$$\mathbf{n}_1 = [\sin(\theta_1)\cos(\phi_1), \sin(\theta_1)\sin(\phi_1), \cos(\theta_1)] \tag{14}$$

$$\mathbf{n}_2 = [\sin(\theta_2)\cos(\phi_2), \sin(\theta_2)\sin(\phi_2), \cos(\theta_2)] \tag{15}$$

$$\mathbf{n}_3 = [\sin(\theta_3)\cos(\phi_3), \sin(\theta_3)\sin(\phi_3), \cos(\theta_3)]. \tag{16}$$

Let us fix $\theta_i = \pi/2, \phi_2 = 2\pi/3, \phi_3 = 4\pi/3$ and vary ϕ_1 from $0 \rightarrow 2\pi$. In the Bloch sphere picture, the states lie on the equator with each state initially equally spaced. ψ_1 or $\mathbf{n}_1$ rotates around once remaining on the equatorial plane while keeping the other two states fixed. Figure 3 shows the anomaly. Since this behaviour is counterintuitive, we will hereafter continue to use the original definitions of P and Q because they avoid the above anomaly and allow a simple relationship between the perimeter and the total overlap. So, in summary, we now have that the larger the perimeter (or equivalently the smaller the Q), the larger the von Neumann entropy keeping all other variables constant.

The geometric phase is a phase that is observed when a state evolves in parameter space (e.g., the parameter could be a magnetic field strength) [5]. A more amenable interpretation for our present purposes is the quantum version of the Pancharatnam relative phase [14]. See [15] for a concise modern introduction. We can think of the trajectory to be specified by a series of projection measurements. For example, if we begin with the state $|\psi_1\rangle$, we can project with $|\psi_2\rangle\langle\psi_2|$, $|\psi_3\rangle\langle\psi_3|$ and back to $|\psi_1\rangle\langle\psi_1|$. Note that the state does not follow any specific trajectory between the three states; we simply have a projective measurement collapsing one state to the subsequent one. However, there is an experimental interpretation of our mixed state geometric phase given in section 4. We can then calculate the geometric phase γ_{ijk} for arbitrary states by

$$\gamma_{ijk} = \arg\{\mathrm{Tr}(|\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j|\psi_k\rangle\langle\psi_k|)\}. \tag{17}$$

It is clear from the above that the states have to be non-orthogonal for this construction to be finite (non-zero) in the first place. Otherwise, no geometric phase is observable. For three states in two dimensions, we get

$$\tan\gamma_{123} = \frac{\mathbf{n}_1 \times \mathbf{n}_2 \cdot \mathbf{n}_3}{1 + \mathbf{n}_1 \cdot \mathbf{n}_2 + \mathbf{n}_2 \cdot \mathbf{n}_3 + \mathbf{n}_3 \cdot \mathbf{n}_1} \tag{18}$$

where $\mathbf{n}_1 \times \mathbf{n}_2$ is the ordinary cross product. For three states in three dimensions, we get [16]

$$\tan\gamma_{123} = \frac{2\sqrt{3}\mathbf{n}_1 \cdot \mathbf{n}_2 \wedge \mathbf{n}_3}{(\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3)^2 + 2\mathbf{n}_1 \cdot \mathbf{n}_2 \star \mathbf{n}_3 - 2} \tag{19}$$

where $\mathbf{n}_1 \cdot \mathbf{n}_2 \wedge \mathbf{n}_3 = n_{1i} f_{ijk} n_{2j} n_{3k}$ and $\mathbf{n}_1 \cdot \mathbf{n}_2 \star \mathbf{n}_3 = \sqrt{3} n_{1i} d_{ijk} n_{2j} n_{3k}$. i, j, k refer to the components of the vectors, f_{ijk} are the antisymmetric $SU(3)$ structure constants and d_{ijk} are the symmetric tensors. Note that γ_{123} refers to the phase taking three states of any dimensionality. Exact definitions and other useful formulae are given in [16, 17] but for convenience, we state them here:

$$[\lambda_i, \lambda_j] = 2i f_{ijk}\lambda_k \tag{20}$$

$$\{\lambda_i, \lambda_j\} = \tfrac{4}{3}\delta_{ij} + 2d_{ijk}\lambda_k \tag{21}$$

$$f_{123} = 1, \quad f_{458} = f_{678} = \tfrac{\sqrt{3}}{2}, \quad f_{147} = f_{246} = f_{257} = f_{345} = f_{516} = f_{637} = \tfrac{1}{2} \tag{22}$$

$$d_{118} = d_{228} = d_{338} = -d_{888} = \tfrac{1}{\sqrt{3}}, \qquad d_{448} = d_{558} = d_{668} = d_{778} = -\tfrac{1}{2\sqrt{3}} \tag{23}$$

$$d_{146} = d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = \tfrac{1}{2}. \tag{24}$$

Other useful formulae are

$$\lambda_i\lambda_j = \tfrac{2}{3}\delta_{ij} + (d_{ijk} + i f_{ijk})\lambda_k \tag{25}$$

$$\mathrm{Tr}\,\lambda_i = 0, \qquad \mathrm{Tr}(\lambda_i\lambda_j) = 2\delta_{ij}. \tag{26}$$

The visibility is defined by

$$V_{ijk} = |\mathrm{Tr}(|\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j|\psi_k\rangle\langle\psi_k|)| \tag{27}$$

where the name originates from V corresponding to how visible or how large the amplitude is in an interferometer [18, 19]. Note that $V\cos\gamma$ is equal to the denominator of $\tan\gamma$ given in equations (18) and (19) for two and three dimensions, respectively. Observe also that $V_{ijk}\cos\gamma_{ijk}$ is the real part of the Bargmann invariant $\langle\psi_i|\psi_j\rangle\langle\psi_j|\psi_k\rangle\langle\psi_k|\psi_i\rangle$ [20, 21]. Bargmann introduced this quantity to distinguish between linear and anti-linear mappings, and employed it in a proof of Wigner's theorem: a symmetry operation on a quantum system is induced by a unitary or an anti-unitary transformation (discrete evolutions allow anti-unitary transformations). So, we could use the Bargmann invariant instead of the geometric phase in our work; however, we will continue using the latter since it is much more familiar to physicists. An interesting avenue for further work will be to explore the further relevance of Bargmann invariants [3] in the following results.

3. Results

In this section we obtain the following results. We first show that in two dimensions, the entropy depends on either the perimeter or the product of the visibility and the cosine of the geometric phase but not both together. We next show that for three states in three dimensions, we need both quantities. The same is shown to be true for three states in any dimension as expected. Then we show that for many states in three dimensions, the entropy depends now on the perimeter and all the possible combinations of the product of the visibility and the cosine of the geometric phase for every triplet of states. In the last subsection, we generalize to any dimensions and any number of states by using a closed-form solution of the entropy obtained by Chumakov *et al* [22]. Note that for general mixtures of pure states (i.e., unequal probabilities), we must redefine the perimeter and geometric phase in two dimensions, and we can no longer express the entropy with just the perimeter, visibility and the cosine of the geometric phase in higher dimensions even after redefinitions.

3.1. Any number of states in two dimensions

As is shown in [11], entropy cannot be increased by increasing the average overlap of the ensemble in two dimensions for any number of states. We will show this by giving an explicit formula of von Neumann entropy as a function of perimeter. We can also rewrite it as a function of geometric phase (modulated by the visibility) but the three quantities will not appear together in the function. We must first find the eigenvalues $x_\pm$ of the density operator which will give $S_{\mathrm{vN}} = -x_+ \ln x_+ - x_- \ln x_-$. Introduce $\mathbf{n} = \frac{1}{t}(\mathbf{n}_1 + \mathbf{n}_2 + \cdots + \mathbf{n}_t) = (n_1, n_2, n_3)$ where $n_i = \frac{1}{t}(n_{1i} + n_{2i} + n_{3i} + \cdots + n_{ti})$. The first subscript refers to the state, the second subscript refers to the vector component and t denotes the number of states. The eigenvalues are $x_\pm = \left(1 \pm \sqrt{n_1^2+n_2^2+n_3^2}\right)/2$. Note that $n_1^2 + n_2^2 + n_3^2 = \mathbf{n}\cdot\mathbf{n}$. Since $\mathbf{n}\cdot\mathbf{n} = (t + 2\mathbf{n}_1\cdot\mathbf{n}_2 + 2\mathbf{n}_2\cdot\mathbf{n}_3 + 2\mathbf{n}_3\cdot\mathbf{n}_1 + \cdots + 2\mathbf{n}_{t-1}\cdot\mathbf{n}_t)/t^2$ and generalizing the definition of P above to t states (6 becomes $t(t-1)$), $\mathbf{n}\cdot\mathbf{n} = (t^2 - P)/t^2$. This gives

$$S_{\mathrm{vN}} = -\frac{1+\sqrt{\frac{t^2-P}{t^2}}}{2}\ln\frac{1+\sqrt{\frac{t^2-P}{t^2}}}{2} - \frac{1-\sqrt{\frac{t^2-P}{t^2}}}{2}\ln\frac{1-\sqrt{\frac{t^2-P}{t^2}}}{2}. \tag{28}$$

Figure 4 plots this. We see that the von Neumann entropy is a monotonically increasing function of perimeter. Using equation (18), we can also write $P = 8 - 2V_{123}\cos\gamma_{123}$ for $t = 3$ where $V_{123}\cos\gamma_{123} = 1 + \mathbf{n}_1\cdot\mathbf{n}_2 + \mathbf{n}_2\cdot\mathbf{n}_3 + \mathbf{n}_3\cdot\mathbf{n}_1$. We find that in two dimensions, increasing the geometric phase corresponds to an increase in entropy (negative values of $\cos\gamma$ become unphysical since at most $P = 9$ and this corresponds to all three states being orthogonal which is not possible in two dimensions). Likewise, decreasing V corresponds to an increase in P and hence entropy. For larger number of states, we cannot generally define the perimeter solely as a function of $V_{ijk}\cos\gamma_{ijk}$, i.e., there will be Bloch vector combinations left over. However, we will indicate in the last subsection that expressing the entropy with the geometric phase for more than three states is possible. It is the Bloch vector picture that makes the explicit calculation unwieldy.

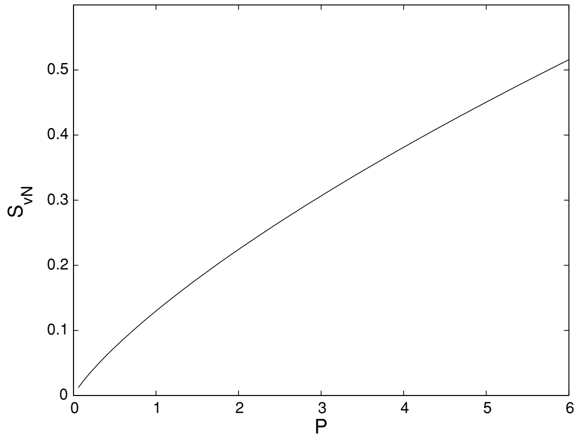

Figure 4. von Neumann entropy versus perimeter in two dimensions for $t = 3$.

The above calculations were for equal probabilities $(p = 1/t)$. If we consider instead unequal probabilities p_i, we must redefine the perimeter:

$$\tilde{P} = |p_1\mathbf{n}_1 - p_2\mathbf{n}_2|^2 + |p_2\mathbf{n}_2 - p_3\mathbf{n}_3|^2 + |p_3\mathbf{n}_3 - p_1\mathbf{n}_1|^2 + \cdots + |p_t\mathbf{n}_t - p_{t-1}\mathbf{n}_{t-1}|^2 \tag{29}$$

and then we obtain

$$\mathbf{n}\cdot\mathbf{n} = t\sum_{j=1}^{t} p_j^2 - \tilde{P} \tag{30}$$

and therefore entropy is still a function of the perimeter albeit with the probabilities. We can see that this reduces to the equal probability result above. For pure states $(p_i = 1)$ the perimeter is zero and $\mathbf{n}\cdot\mathbf{n} = 1$ thus $S_{\mathrm{vN}} = 0$, as it should be.

Note that in the case of unequal probabilities, there is no straightforward method of relating the perimeter to the geometric phase unless we redefine the geometric phase by incorporating the unequal probabilities. We can, for example, consider three states with $\rho_i = \frac{1}{2}(1 + p_i\mathbf{n}_i\cdot\sigma)$ where i corresponds to the ith state. On the Bloch sphere this state points in the same direction as the original pure state, but it is shortened by p_i. This yields

$$V_{123}\cos\gamma_{123} = 1 + p_1p_2\mathbf{n}_1\cdot\mathbf{n}_2 + p_2p_3\mathbf{n}_2\cdot\mathbf{n}_3 + p_1p_3\mathbf{n}_1\cdot\mathbf{n}_3 \tag{31}$$

which can be related to $\tilde{P}$ for $t = 3$. Once this is done, of course, we can no longer speak about the state evolving into each other by projective measurement since they are no longer pure.

We should comment that the geometric phase is dependent on the decomposition of the density matrix whereas the von Neumann entropy is independent of the decomposition. This means that we cannot express the von Neumann entropy in terms of the geometric phase for all the decompositions of a given density matrix. A simple example is the following. A three state decomposition of a density matrix yields an entropy expressed in terms of the geometric phase but if the decomposition of the same density matrix contains only two states, it is not possible (we need at least three states to express the von Neumann entropy as a function of geometric phase).

3.2. Three states in three dimensions

Similar steps are taken as the previous subsection. First, we introduce the coherence vector $\mathbf{n} = (n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8) = \frac{1}{3}(\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3)$ with equal probabilities. The perimeter is given by $\mathbf{n} \cdot \mathbf{n} = \frac{1}{9}(9 - P)$. Our density matrix is now a 3×3 matrix, and in order to compute its entropy we need to be able to find the eigenvalues first. This leads to solving the following cubic equation:

$$x^3 + Ax^2 + Bx + C = 0 \tag{32}$$

where

$$A = -1 \tag{33}$$

$$B = \frac{1 - \mathbf{n} \cdot \mathbf{n}}{3} \tag{34}$$

$$C = \frac{\mathbf{n} \cdot \mathbf{n}}{9} - \frac{1}{27} - \frac{2}{27} V_{123} \cos \gamma_{123} \tag{35}$$

with $V_{123} \cos \gamma_{123} = (\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3)^2 + 2\mathbf{n}_1 \cdot \mathbf{n}_2 \star \mathbf{n}_3 - 2$. The solution is given by [23, 24]

$$\begin{aligned} x_1 &= 2\sqrt{-T} \cos \frac{\theta}{3} - \frac{A}{3} \\ x_2 &= 2\sqrt{-T} \cos \frac{\theta + 2\pi}{3} - \frac{A}{3} \\ x_3 &= 2\sqrt{-T} \cos \frac{\theta + 4\pi}{3} - \frac{A}{3} \end{aligned} \tag{36}$$

where

$$R = \frac{9AB - 27C - 2A^3}{54} = \frac{V_{123} \cos \gamma_{123}}{27} \tag{37}$$

$$T = \frac{3B - A^2}{9} = -\frac{\mathbf{n} \cdot \mathbf{n}}{9} \tag{38}$$

$$\theta = \arccos \frac{R}{\sqrt{-T^3}} = \arccos \frac{V_{123} \cos \gamma_{123}}{(\mathbf{n} \cdot \mathbf{n})^{3/2}}. \tag{39}$$

The von Neumann entropy is

$$S_{\mathrm{vN}} = -x_1 \ln x_1 - x_2 \ln x_2 - x_3 \ln x_3. \tag{40}$$

Let us look at a couple of examples. The first example uses the three states given in [16]:

$$|\psi_1\rangle = |2\rangle \tag{41}$$

$$|\psi_2\rangle = \sin \xi |1\rangle + \cos \xi |2\rangle \tag{42}$$

$$|\psi_3\rangle = \sin \eta \cos \zeta |0\rangle + \mathrm{e}^{\mathrm{i}\chi} \sin \eta \sin \zeta |1\rangle + \cos \eta |2\rangle \tag{43}$$

where $0 \leqslant \xi, \eta, \zeta \leqslant \pi/2$ and $0 \leqslant \chi < 2\pi$. By setting $\xi = \pi/2$, $\chi = 0$ and $\eta = \pi/2$, we can set the geometric phase γ and visibility V to vanish. Then by varying ζ, we can observe the dependence of von Neumann entropy S_{vN} on perimeter P between $6 \leqslant P \leqslant 9$ as is shown in figure 5. As is the case in two dimensions (figure 4), S_{vN} increases monotonically when P increases. Note that S_{vN} is bounded by the maximum entropy allowable in a d-dimensional system $S_{\max} = \ln d$. In fact the monotonically increasing property can be explicitly checked

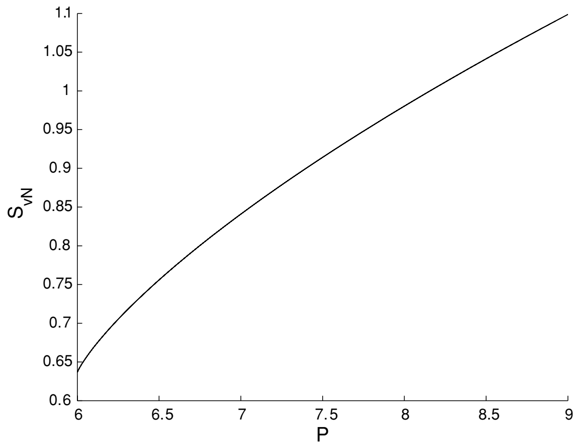

Figure 5. von Neumann entropy versus perimeter in three dimensions. We see entropy monotonically increasing when perimeter is increased as is the case in two dimensions.

by differentiating equation (40) with respect to P and realizing the result to be positive for the above range of P. In order to inspect smaller values of P, we must also vary V or γ. Note that finding three states to vary in order to change the geometric phase alone is nontrivial. The next example does exactly that.

We choose the three states given in [11]:

$$|\psi_1\rangle = |0\rangle \tag{44}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \tag{45}$$

$$|\psi_3\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{2\,\mathrm{e}^{\mathrm{i}\gamma} - 1}{\sqrt{3}}|1\rangle + \sqrt{\frac{4}{3}\cos\gamma - 1}|2\rangle \tag{46}$$

where γ is the geometric phase which is bounded here by $\gamma_{\max} = 0.72$ radians. These three states keep V and P fixed so that we can inspect how S_{vN} depends on γ alone. By calculating the perimeter, geometric phase and visibility and using equation (40), we obtain figure 6 which is identical to the graph given in [11].

In contrast to the two-dimensional case where we can only increase/decrease the entropy as we increase/decrease the geometric phase, we also have that the entropy decreases/increases as the geometric phase increases/decreases. Although this is only one particular example, it is counterintuitive. We can deform the states slightly so as to increase the sum of overlaps Q (or decrease the perimeter P), hence decrease the distinguishability. This should decrease the entropy but we can now compensate by decreasing the geometric phase sufficiently to provide an overall increase in entropy [11].

Why should there be a transition between the two- to the three-dimensional case? Mathematically speaking, the distinctions are that the symmetric tensor does not exist in two dimensions and the homomorphism between $SU(2)$ and $SO(3)$ does not exist between $SU(3)$ and $SO(8)$. This means that in the eight-dimensional ball, there are patches corresponding to unphysical states and therefore our intuition of what the phase is geometrically as well as how the perimeter changes is lost. Alternatively, one could perhaps think in terms of the degree

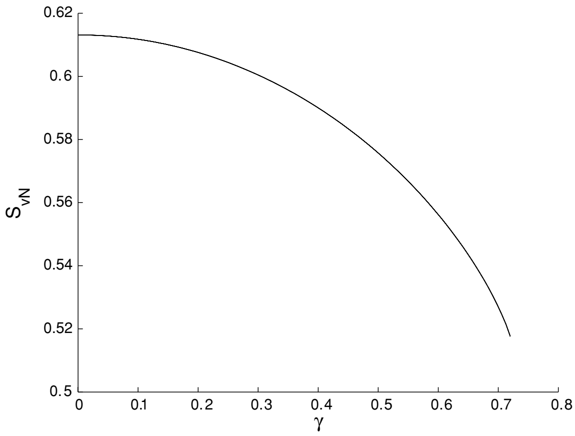

Figure 6. von Neumann entropy versus geometric phase in three dimensions. We clearly see that as the geometric phase is increased, von Neumann entropy decreases in contrast to the opposite behaviour found in two dimensions. The result shows for equal probabilities of states but the monotonically decreasing behaviour can also be found with unequal probabilities.

of freedom required to find the eigenvalues of the density matrix. In two dimensions, there is a single degree of freedom which is why we only need either the perimeter or the geometric phase. In three dimensions, we have two degrees of freedom which are the perimeter and the geometric phase. A precise formulation of this remark is left for future research.

3.3. Three states in any dimensions

We can increase the number of dimensions arbitrarily by considering three general states $|\alpha\rangle, |\beta\rangle$ and $|\delta\rangle$. We can construct the density operator with equal probabilities, $\rho = \frac{1}{3}(|\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta| + |\delta\rangle\langle\delta|)$. As the above states are not orthogonal, we use the Gram–Schmidt procedure to obtain the following orthogonal states:

$$|v_1\rangle = \frac{|\alpha\rangle}{\||\alpha\rangle\|} = |\alpha\rangle \tag{47}$$

$$|v_2\rangle = \frac{|\beta\rangle - \langle v_1|\beta\rangle|v_1\rangle}{\||\beta\rangle - \langle v_1|\beta\rangle|v_1\rangle\|} = \frac{|\beta\rangle - \langle v_1|\beta\rangle|v_1\rangle}{\sqrt{1-|\langle\alpha|\beta\rangle|^2}} \tag{48}$$

$$|v_3\rangle = \frac{|\delta\rangle - \langle v_2|\delta\rangle|v_2\rangle - \langle v_1|\delta\rangle|v_1\rangle}{\||\delta\rangle - \langle v_2|\delta\rangle|v_2\rangle - \langle v_1|\delta\rangle|v_1\rangle\|} = \frac{|\delta\rangle - \langle v_2|\delta\rangle|v_2\rangle - \langle v_1|\delta\rangle|v_1\rangle}{\sqrt{1-|\langle v_2|\delta\rangle|^2 - |\langle\alpha|\delta\rangle|^2}}. \tag{49}$$

We can invert these and substitute into the density operator. By noting that $Q = |\langle\alpha|\beta\rangle|^2 + |\langle\beta|\delta\rangle|^2 + |\langle\delta|\alpha\rangle|^2$ (used instead of perimeter) and $V\cos\gamma = \Re\{\langle\alpha|\delta\rangle\langle\delta|\beta\rangle\langle\beta|\alpha\rangle\}$ we find that

$$A = -1 \tag{50}$$

$$B = \frac{3-Q}{9} \tag{51}$$

$$C = \frac{-1+Q-2V_{123}\cos\gamma_{123}}{27} \tag{52}$$

which give

$$R = \frac{V_{123}\cos\gamma_{123}}{27} \tag{53}$$

$$T = -\frac{Q}{27} \tag{54}$$

$$\theta = \arccos\frac{\sqrt{27}V_{123}\cos\gamma_{123}}{Q^{3/2}}. \tag{55}$$

By noting that $\mathbf{n}\cdot\mathbf{n} = 1 - P/9$ and $Q = (9-P)/3$ gives $\mathbf{n}\cdot\mathbf{n} = Q/3$, we find that these equations are identical to the equivalent ones appearing in the previous subsection. This is not surprising because three states in arbitrary dimensions can be represented by a rank three matrix. The important point to observe is that we do not require any other quantity to define entropy. We still only require the total overlap (or perimeter), geometric phase and visibility.

For unequal probabilities, we do not get a simple generalization as in the two-dimensional case. We have explicitly

$$A = -1 \tag{56}$$

$$B = p_1p_2 + p_2p_3 + p_1p_3 - p_1p_2|\langle\alpha|\beta\rangle|^2 - p_2p_3|\langle\beta|\delta\rangle|^2 - p_1p_3|\langle\alpha|\delta\rangle|^2 \tag{57}$$

$$C = p_1p_2p_3(-1 + Q - 2V_{123}\cos\gamma_{123}). \tag{58}$$

Observe that as well as Q and $V_{123}\cos\gamma_{123}$, we also have individual overlaps in B which cannot be written in terms of Q, V_{123} or γ_{123}. They reduce to the equal probabilities case and for a pure state, $B = C = 0$ therefore when substituting into equation (36), we obtain the desired $S_{\mathrm{vN}} = 0$. It is interesting to note that now R (equation (37)) contains the overlap as well as the visibility and geometric phase, hence altering the above form of entropy. Since the probabilities directly influence how mixed the ensemble is, it is not surprising that the form of the entropy should change.

3.4. Any number of states in three dimensions

We have so far looked at only three states in effectively three dimensions. We will now consider the three-dimensional case with N number of states. We now have $\mathbf{n} = (n_1, n_2, \ldots, n_8) = \frac{1}{N}(\mathbf{n}_1 + \mathbf{n}_2 + \cdots + \mathbf{n}_{N-1} + \mathbf{n}_N)$. The perimeter can be written in a compact form as

$$P = N(N-1) - 2\sum_{i>j=1}^{N}\mathbf{n}_i\cdot\mathbf{n}_j. \tag{59}$$

Note that this is also true for any dimensions. Another useful formula is

$$\mathbf{n}\cdot\mathbf{n} = \frac{1}{N^2}\left[N + 2\sum_{i>j=1}^{N}\mathbf{n}_i\cdot\mathbf{n}_j\right] \tag{60}$$

also true for any dimensions. We know that the cubic coefficients A and B remain the same as before. C is the only one that needs to be modified. We find that

$$C = \frac{\mathbf{n}\cdot\mathbf{n}}{9} - \frac{1}{27} - \frac{2}{N^3}\sum_{i>j>k=1}^{N}V_{ijk}\cos\gamma_{ijk} + \frac{2}{9N^3}\left(\frac{N!}{(N-3)!3!} + (N-3)(N(N-1)-P) - \frac{N}{3}\right). \tag{61}$$

Note that when $N = 3$, this reduces to the aforementioned three-state case. It is interesting to note that the geometric phase term still contains only three states albeit with all the possible combinations of three states. This is perhaps contrary to what we might expect from the explanation offered in the introduction which suggests that all the states should exist in a single geometric phase. However, we will show in the next subsection that it is possible to express the entropy in terms of the geometric phase with more than three states.

3.5. Any number of states in any dimension

We know that the von Neumann entropy can be expanded in a power series $S_{\mathrm{vN}} = \sum_{i=1}^{\infty} c_i \,\mathrm{Tr}\,\rho^i$ where c_i are the expansion coefficients. The exact values of the c_is are not relevant for our discussion. Keyl and Werner [25] have shown that in order to calculate the eigenvalues of a d-dimensional density matrix, it is necessary and sufficient to obtain all the traces of the powers of the density matrix up to the dth power. $\mathrm{Tr}\,\rho^2$ contains the perimeter and $\mathrm{Tr}\,\rho^3$ contains the geometric phase with three states as is shown in the next section as well as below. With a d-dimensional system, the entropy will contain geometric phase terms up to d states. Obtaining a closed-form solution of the entropy for higher than four dimensions is difficult because there is no equation using only radicals to solve the quintic or higher equation. However, Chumakov *et al* [22] have a closed-form solution for arbitrary dimensional systems which requires traces of powers of the density matrices up to $d - 1$ (note that this is one less trace than in [25], achieved by using the determinant and other combinations of the elements of the density matrix). Using their result, it is possible to give a closed-form solution of the von Neumann entropy in terms of the geometric phase but we will only show that for higher dimensions, there still exist geometric phase terms.

These traces contain the sum of overlaps Q (or equivalently perimeter) and all combinations of the product of visibility and the cosine of the geometric phase up to d states, e.g., $V_{ab\ldots d}\cos\gamma_{ab\ldots d}$. Below, we give explicitly the form of the traces up to fourth dimension of a density matrix $\rho = \frac{1}{N}\sum_{i=1}^{N}\rho_i$ with equal mixture of N states.

$$\mathrm{Tr}(\rho^2) = \frac{1}{N^2}(N + 2Q) \tag{62}$$

$$\mathrm{Tr}(\rho^3) = \frac{1}{N^3}\left(N + 6Q + 6\sum_{i<j<k}^{N} V_{ijk}\cos\gamma_{ijk}\right) \tag{63}$$

$$\begin{aligned}\mathrm{Tr}(\rho^4) = \frac{1}{N^4}\Bigg(&N + 6Q + \sum_{i\neq j}^{N}\mathrm{Tr}(\rho_i\rho_j\rho_i\rho_j) + 2\sum_{i\neq j\neq k}^{N}\mathrm{Tr}(\rho_i\rho_j\rho_k\rho_j)\\ &+ 24\sum_{i<j<k}^{N} V_{ijk}\cos\gamma_{ijk} + 8\sum_{i<j\neq k\neq l}^{N} V_{ijkl}\cos\gamma_{ijkl}\Bigg).\end{aligned} \tag{64}$$

We can observe again that for two dimensions, the entropy can be expressed in terms of Q and for three dimensions, it can be written in terms of Q and $V_{ijk}\cos\gamma_{ijk}$. However, in four dimensions, although there are terms of Q and the geometric phase, there are also terms that involve powers of overlaps and products of different overlaps. So we conclude that even for higher dimensional systems, the entropy can be expressed as a function of perimeter and the product of visibility and the cosine of the geometric phases, although other terms involving overlaps become relevant. We expect similar expressions for traces of higher powers of the density matrix.

We would like to briefly mention how we can incorporate geometric phases of more than three states in the three-dimensional case with the density matrix decomposition containing more than three states (the same reasoning applies to the two-dimensional case). We only encountered three states in the geometric phase because we effectively only went up to $\mathrm{Tr}\,\rho^3$. Together with $\mathrm{Tr}\,\rho^2$ and $\mathrm{Tr}\,\rho$, we are able to find the three eigenvalues (e_1, e_2 and e_3) of the density matrix. Namely, $e_1+e_2+e_3 = 1$, $e_1^2+e_2^2+e_3^2 = \mathrm{Tr}\,\rho^2$ and $e_1^3+e_2^3+e_3^3 = \mathrm{Tr}\,\rho^3$. Instead of $\mathrm{Tr}\,\rho^3$, we could use $\mathrm{Tr}\,\rho^4 = e_1^4 + e_2^4 + e_3^4$ to find all three eigenvalues. This will yield a geometric phase with more than three states as can be seen from equation (64). Similarly, we could use instead $\mathrm{Tr}\,\rho^k$ where k is an integer greater than four. This should incorporate into the von Neumann entropy, geometric phase terms involving k states.

4. Experimental measurements of entropy, perimeter and phase

We use the simple quantum network based on the controlled-swap gate presented in [18] which extracts properties of quantum states bypassing the need for quantum tomography. Physically, the network is a representation of the Mach–Zehnder interferometer [26]. This set-up was introduced to define the mixed state geometric phase. We show below that the same set-up can be used to calculate the von Neumann entropy, hence we relate the entropy and geometric phase experimentally. This is more evidence that the relationship between entropy and geometric phase is not unexpected.

Since we have shown the von Neumann entropy as a function of perimeter (overlap), geometric phase and visibility, we can experimentally measure this entropy by calculating $\mathrm{Tr}(\rho^2)$ for the perimeter and $\mathrm{Tr}(\rho^3)$ for the visibility and geometric phase where $\rho = \frac{1}{3}(\rho_1 + \rho_2 + \rho_3)$ with equal probabilities for three states in three dimensions. However, we can generalize this experimental procedure for any dimensions and any number of states by calculating the traces of up to the dth power of the density matrix $\rho = \frac{1}{N}\sum_{i=1}^{N}\rho_i$ where N is the number of states. Consider a set-up with two separable subsystems $\rho\otimes\rho$ and three separable subsystems $\rho\otimes\rho\otimes\rho$. We now introduce the swap operator W, $W|a\rangle\otimes|b\rangle = |b\rangle\otimes|a\rangle$ and the shift operator F, $F|a\rangle\otimes|b\rangle\otimes|c\rangle = |c\rangle\otimes|a\rangle\otimes|b\rangle$ for any pure states $|a\rangle$, $|b\rangle$ and $|c\rangle$. The experimental procedure which will be described shortly measures $\mathrm{Tr}\,W(\rho\otimes\rho) = \mathrm{Tr}(\rho^2)$ [18] and similarly $\mathrm{Tr}\,F(\rho\otimes\rho\otimes\rho) = \mathrm{Tr}(\rho^3)$. This can be readily generalized to the dth power of ρ using the general shift operator S where $S|a\rangle\otimes\cdots\otimes|c\rangle\otimes|d\rangle = |d\rangle\otimes|a\rangle\cdots\otimes|c\rangle$ so that $\mathrm{Tr}\,S(\rho^{\otimes d}) = \mathrm{Tr}(\rho^d)$. We find on expansion:

$$\mathrm{Tr}\,\rho^2 = \tfrac{1}{9}(3 + 2\,\mathrm{Tr}\,\rho_1\rho_2 + 2\,\mathrm{Tr}\,\rho_2\rho_3 + 2\,\mathrm{Tr}\,\rho_1\rho_3) = \tfrac{1}{9}(3 + 2Q) \tag{65}$$

$$\begin{aligned}\mathrm{Tr}\,\rho^3 &= \tfrac{1}{27}(3 + 6\,\mathrm{Tr}\,\rho_1\rho_2 + 6\,\mathrm{Tr}\,\rho_2\rho_3 + 6\,\mathrm{Tr}\,\rho_1\rho_3 + 3\,\mathrm{Tr}\,\rho_1\rho_2\rho_3 + 3\,\mathrm{Tr}\,\rho_1\rho_3\rho_2)\\ &= \tfrac{1}{27}(3 + 6Q + 6V_{123}\cos\gamma_{123}).\end{aligned} \tag{66}$$

The last line follows from $\mathrm{Tr}\,\rho_1\rho_2\rho_3 = V_{123}\,\mathrm{e}^{\mathrm{i}\gamma_{123}}$ and $\mathrm{Tr}\,\rho_1\rho_3\rho_2 = V_{123}\,\mathrm{e}^{-\mathrm{i}\gamma_{123}}$. Hence, on obtaining Q and $V_{123}\cos\gamma_{123}$, we can calculate S_{vN} for three states in three dimensions. In principle, we can also expand $\mathrm{Tr}\,\rho^d$ to show that it contains Q and all the combinations of the product of visibility and the cosine of the geometric phase. Figure 7 shows the experimental set-up that may be used to measure the von Neumann entropy (the diagram shows the case for two inputs of ρ but for a rank d density matrix, we must inspect up to d inputs of ρ). We will briefly describe how it calculates $\mathrm{Tr}\,\rho^2$, and then $\mathrm{Tr}\,\rho^d$ is a straightforward extension.

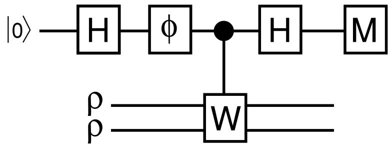

Figure 7. Experimental set-up to ascertain $\operatorname{Tr}\rho^2$. We must exchange the swap operator W with a shift operator S and input d ρs to obtain $\operatorname{Tr}\rho^d$.

We begin with the initial state $\rho_{\rm in} = |0\rangle\langle 0| \otimes \rho \otimes \rho$. We apply the first Hadamard gate $H = \frac{1}{2}\left(\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right)$:

$$\rho_H = (H \otimes \mathbb{I} \otimes \mathbb{I})(|0\rangle\langle 0| \otimes \rho \otimes \rho)(H^\dagger \otimes \mathbb{I} \otimes \mathbb{I}) \tag{67}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \otimes \rho \otimes \rho. \tag{68}$$

Then we apply the phase shift $\Phi = \left(\begin{smallmatrix}e^{i\phi} & 0\\ 0 & 1\end{smallmatrix}\right)$ to get $\rho_\Phi = \frac{1}{2}\left(\begin{smallmatrix}1 & e^{-i\phi}\\ e^{i\phi} & 1\end{smallmatrix}\right) \otimes \rho \otimes \rho$. Next is the controlled-swap operation:

$$U_{\rm cs} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \otimes \mathbb{I} \otimes \mathbb{I} + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \otimes W \tag{69}$$

and finally another Hadamard to obtain

$$\rho_{\rm out} = \frac{1}{4}\left[\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \otimes \rho \otimes \rho + \begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} \otimes W(\rho \otimes \rho)W^\dagger \right.$$
$$\left. + e^{i\phi}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix} \otimes (\rho \otimes \rho)W^\dagger + e^{-i\phi}\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix} \otimes W(\rho \otimes \rho)\right]. \tag{70}$$

Since measuring the intensity I is proportional to the probability, we can measure in the computational basis $|0\rangle$ to get

$$I \propto \operatorname{Tr}[|0\rangle\langle 0| \otimes \mathbb{I} \otimes \mathbb{I}\rho_{\rm out}]$$
$$\propto \operatorname{Tr}\rho\operatorname{Tr}\rho + \operatorname{Tr}(W\rho \otimes \rho W^\dagger) + e^{i\phi}\operatorname{Tr}(\rho \otimes \rho W^\dagger) + e^{-i\phi}\operatorname{Tr}(W\rho \otimes \rho)$$
$$= 1 + 1 + e^{i\phi}[\operatorname{Tr} W\rho \otimes \rho]^* + e^{-i\phi}\operatorname{Tr} W\rho \otimes \rho \tag{71}$$

$$= 2 + e^{i\phi}|\operatorname{Tr}\rho^2|\, e^{-i\arg\operatorname{Tr}\rho^2} + e^{-i\phi}|\operatorname{Tr}\rho^2|\, e^{i\arg\operatorname{Tr}\rho^2} \tag{72}$$

$$= 2 + 2|\operatorname{Tr}\rho^2|\cos[\phi - \arg\operatorname{Tr}\rho^2]. \tag{73}$$

We are able to adjust the phase ϕ so as to obtain the largest intensity yielding $|\operatorname{Tr}\rho^2|$ and $\phi = \arg\operatorname{Tr}\rho^2$. Then, we acquire $\operatorname{Tr}\rho^2 = |\operatorname{Tr}\rho^2|\, e^{i\arg\operatorname{Tr}\rho^2}$. Note that $\operatorname{Tr}\rho^2$ is real but higher powers of ρ are generally not. We also obtain $\operatorname{Tr}\rho^3$ following similar steps. We can obtain the von Neumann entropy via (65) and (66) for three dimensions. Naturally, we can calculate the von Neumann entropy for d-dimensional systems by calculating the trace of the powers of ρ up to $d - 1$ and utilizing the formula given in [22]. So we see that the set-up in figure 7 allows us to measure both the entropy and the product of the visibility and the cosine of the geometric phase.

5. Summary

We have explicitly shown the dependence of entropy on the perimeter, geometric phase and the visibility. For an arbitrary number of states in the two-dimensional case, entropy is solely a function of perimeter whereas for three states in three dimensions and more states in higher dimensions, entropy is no longer just a function of perimeter but also of geometric phase and visibility. Finally, we have shown a possible way to obtain the von Neumann entropy experimentally. The same experimental interferometric set-up can also be used to measure the visibility and geometric phase associated with a set of pure states. This clarifies why physically the two seemingly unrelated concepts of entropy and geometric phase should in fact be related to each other.

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